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TECHNICAL NOTE

Incremental heat transfer number in the entry region of circular tubes

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1. INTRODUCTION

THE THERMAL entry region in ducts has been studied extensively in the past, see Shah and London [1]. Many investigators have analyzed the hydrodynamic entry length problem, for example see Vrentas *et al.* [2] and Christiansen *et al.* [3], and for the thermally developing flow with a developed velocity profile, see Hennecke [4], Hsu [5], and Verhoff and Fisher [6]. Recently, Nguyen [7] applied the finite difference method to this problem and provided accurate values for the incremental heat transfer number and entrance length for *Pe* ranging from 1 to 1000.

The simultaneously developing flow in a circular tube, where both the velocity and temperature profiles are developing together, was first analyzed by Kays [8] in 1955. Shah and London reviewed most of the relevant analyses and tabulated *Nu* results from Hornbeck [9], Manohar [10], and Hwang [11]. Pagliarini's [12] is the only study found in the literature of the axial diffusion effects in simultaneously developing flow in a circular tube with infinite extent.

In this paper, the results of a numerical study of the simultaneously developing flow in a circular tube, accounting for axial diffusion of momentum and heat, are presented. The velocity and temperature profiles are assumed to be uniform at upstream infinity. Two types of thermal boundary conditions are considered—a constant axial wall temperature and a constant wall heat flux. The velocities and temperature are calculated using a finite difference scheme on a stretched mesh. Extended Richardson extrapolation is used to extrapolate three mesh sizes to zero mesh size giving excellent agreement with previous accurate solutions, see Nguyen [13] and Nguyen and Maclaine-cross [14].

2. FORMULATION OF PROBLEM

In the present study we analyze the steady laminar flow of a Newtonian constant property fluid through a circular tube of infinite extent. Both the temperature T_∞ and velocity U_∞ are uniform at upstream infinity. The upstream region is a stream tube which is impermeable, frictionless, and thermally insulated. The real tube downstream of the entrance is infinite in extent with a fully developed parabolic velocity profile at the exit and subject to the boundary condition of uniform wall temperature or uniform wall heat flux. In the present analysis, the origin of the axes is located on the tube wall. This system of axes is convenient in setting up the solution field and marching the iterative procedure.

The equations of motions are written in terms of the stream function ψ and vorticity ζ, defined by

$$u = \left(\frac{2}{2r-1}\right)\frac{\partial\psi}{\partial r},$$
$$v = -\left(\frac{2}{2r-1}\right)\frac{\partial\psi}{\partial x}, \quad \text{and} \quad \zeta = \frac{\partial u}{\partial r} - \frac{\partial v}{\partial x}, \tag{1}$$

where u, v are velocity components in the x-, r-direction.

The dimensionless governing equations for vorticity, stream function and temperature, derived from the Navier–Stokes equations, are as follows

$$\frac{\partial\zeta}{\partial t} = -u\frac{\partial\zeta}{\partial x} - v\frac{\partial\zeta}{\partial r} + \frac{2}{2r-1}v\zeta + \frac{1}{Re}\left(\frac{\partial^2\zeta}{\partial r^2} + \frac{2}{2r-1}\frac{\partial\zeta}{\partial r} - \frac{4}{(2r-1)^2}\zeta + \frac{\partial^2\zeta}{\partial x^2}\right) \tag{2}$$

$$0 = \left(\frac{\partial^2\psi}{\partial r^2} + \frac{\partial^2\psi}{\partial x^2}\right) - \frac{2}{2r-1}\frac{\partial\psi}{\partial r} - \frac{2r-1}{2}\zeta \tag{3}$$

$$\frac{\partial\theta}{\partial t} = -u\frac{\partial\theta}{\partial x} - v\frac{\partial\theta}{\partial r} + \frac{1}{Re\,Pr}\left(\frac{\partial^2\theta}{\partial r^2} + \frac{2}{2r-1}\frac{\partial\theta}{\partial r} + \frac{\partial^2\theta}{\partial x^2}\right) \tag{4}$$

in which $\theta = (T - T_\infty)/(T_w - T_\infty)$, $Re = U_\infty D_h/\nu$, and $Pr = \nu/\alpha$. Here D_h, ν, and α denote, respectively, the hydraulic diameter of the circular tube, the kinetic viscosity and the thermal diffusivity.

For details on the method of solution and notations, see Nguyen [13] and Nguyen and Maclaine-cross [14].

3. ISOTHERMAL FLOW RESULTS AND PREVIOUS SOLUTIONS

The centerline velocity at the entrance ($x = 0$) for various *Re* is shown in Table 1. At low *Re*, with a uniform inlet condition at upstream infinity, the upstream diffusion of momentum is quite significant as can be seen from the development of the centerline velocity at the entrance. The dimensionless velocity increases with decreasing *Re* at $x = 0$ and is 6.6–61.8% higher than the mean velocity u_m. Inside the stream tube, wall shear takes effect and the centerline velocity increases with *Re*. From the graphical results of Pagliarini [12], the values for the centerline velocity at $Re = 5$ and

Table 1. u_{max}/u_m at entrance ($x = 0$)

$Re = 1$	2	5	10	20	50	100	200	1000
1.6424	1.6175	1.5447	1.4422	1.3145	1.1911	1.1352	1.0973	1.0661

Table 2. Hydrodynamic entrance length L_{hy}^{+}

Re	Present	Pagliarini	Christiansen	Vrentas
1	0.3831	0.385	0.333	0.33
2	0.2025			
5	0.09506	0.095	0.086	
10	0.06301		0.059	
20	0.05553			
50	0.05439	0.054	0.050	0.047
100	0.05462		0.051	0.047
200	0.05483		0.055	0.0508
1000	0.05500	0.055	0.056	

50 are 1.54 and 1.19, respectively, which are in very good agreement with the present solution.

The non-dimensional hydrodynamic entrance length L_{hy}^{+}, defined as the duct length required to achieve a duct section maximum velocity of 99% of the fully developed value (2 for the circular tube), is presented in Table 2 with the available data from Pagliarini [12], Christiansen *et al.* [3], and Vrentas *et al.* [2]. The present L_{hy}^{+} is in excellent agreement with the values obtained by Pagliarini and is longer than the values given in the earlier numerical analyses. $L_{hy}^{+} = 0.056$ and is constant when based on a boundary layer type analysis (Shah and London [1]). However when the complete set of Navier–Stokes equations is solved L_{hy}^{+} is a strong function of *Re* for low *Re* flows, as seen in Table 2.

For $Pr = 0.7$ and $1 \leqslant Re \leqslant 10$, the following correlation is provided to approximate these values with the error ranging from 0.6% at $Re = 2$ to 3.3% at $Re = 10$

$$L_{hy}^{+} = 0.02519 + 0.3572/Re. \tag{5}$$

For $Re \geqslant 20$ L_{hy}^{+} can be taken as 0.055 with the maximum error of 1.1% at $Re = 50$.

Atkinson *et al.* [15] and Friedmann *et al.* [16] presented equations to calculate L_{hy}^{+} for the hydrodynamically developing flow problem with a uniform entrance velocity profile. These equations are a linear combination of creeping flow, obtained by minimization of viscous dissipation using a finite element method, and boundary layer type solution (Atkinson *et al.*), or are based on the solution of the Navier–Stokes equations (Friedmann *et al.*). The values of L_{hy}^{+} given from these equations are considerably higher than those in Table 2. The main contributing factor for this is the significant upstream diffusion of momentum in the case considered here, hence more developed velocity profiles and shorter entrance lengths.

For the case of simultaneously developing flow, Shah and London [1] have presented numerical solutions for the local Nusselt numbers Nu_x from Manohar [10] and Hwang and Sheu [17]. Table 3 compares these with values interpolated from the present solution for $Pr = 0.7$ and $Re = 1000$. The present fully developed flow values agree to five significant figures with classical analytic solutions, whereas Manohar's $Nu_{x,T}$ approaches 3.63 asymptotically, a value 1% lower than the exact value of 3.6568. The present $Nu_{x,H}$ are in good agreement with those of Manohar, but the values of $Nu_{x,T}$ are slightly higher than the previous values.

4. CONSTANT WALL TEMPERATURE RESULTS

The asymptotic Nusselt number for fully developed flow was presented precisely by Shah and London as

$$Nu_T = 3.6567935 \tag{6}$$

for the case of negligible axial heat conduction, viscous dissipation, flow work, and thermal energy sources within the fluid. When the effect of axial heat conduction is included, Nu_T is a strong function of the Péclet number. The results for Nu_T from the present numerical work are presented in Table 4 for comparison with values given in Shah and London. The present Nu_T is only 0.3% lower than the analytic value for $Pe = 0.7$ and agrees to 5 significant figures at higher *Pe*.

In the case of simultaneously developing flow in circular tubes, values for the incremental heat transfer number $N(\infty)$ are non-existent in the literature for comparison with the present work. This is due to the fact that in most previous solutions, the approximation of $N(\infty)$ usually deteriorates at the end of the thermal entrance. In the present work, discretization error is reduced by a combination of efficient numerical schemes, stretched mesh, and the extrapolation to zero mesh size. To show some typical discretization errors, the fully developed incremental heat transfer number for constant wall temperature $N_T(\infty)$ is given in Table 5, to-

Table 3. Comparison of Nu_x for $Pr = 0.7$ and $Re = 1000$

	$Nu_{x,T}$			$Nu_{x,H}$	
x^*	Present	Manohar	Hwang	Present	Manohar
0.0	37.9601	—	—	40.6309	—
0.002857	8.4482	8.24	8.129	11.1313	11.33
0.003571	7.8471	7.54	7.469	10.3739	10.31
0.007143	5.8929	5.84	5.793	7.8012	7.854
0.010710	5.1761	5.11	5.081	6.7458	6.792
0.014290	4.7648	4.69	4.671	6.1629	6.179
0.017860	4.4885	4.42	4.409	5.7663	5.774
0.021430	4.2945	4.23	4.224	5.4807	5.486
0.028570	4.0469	3.998	3.993	5.1018	5.108
0.035710	3.9037	3.846	3.862	4.8690	4.879
0.071430	3.6890	3.641	3.674	4.4523	4.460
0.076610	3.6817	3.632	—	4.4328	4.439
0.107100	3.6620	—	3.655	4.3795	—

Table 4. Fully developed Nu_T as a function of Pe

Re	0.7	1.4	3.5	7	14	35	70	140	700
Present	4.0838	3.9889	3.8264	3.7263	3.6784	3.6611	3.6585	3.6568	3.6568
Ref. [1]	4.071			3.728				3.6568	3.6568

Table 5. Incremental heat transfer number $N_T(\infty)$ for $Pr = 0.7$

Re	11 × 81	21 × 161	41 × 321	Extrapolated	Diff. (%)
1	0.9059	0.9577	1.0397	1.0802	1.2
2	0.4717	0.5101	0.5528	0.5733	1.1
5	0.2071	0.2207	0.2380	0.2464	1.0
10	0.1101	0.1258	0.1367	0.1417	0.9
20	0.08566	0.09065	0.09589	0.09839	0.7
50	0.05441	0.06626	0.07143	0.07357	0.6
100	0.04796	0.06023	0.06559	0.06781	0.6
200	0.04633	0.05680	0.06362	0.06669	1.2
1000	0.04575	0.05602	0.06218	0.06492	1.1

gether with the values for the three grid meshes used in its calculation. The difference between two point and three point extrapolation is given in the last column and the largest difference is 1.2%. It is believed that the residual discretization error in the three point extrapolations used in this work is less than 1.2%.

For $Pr = 0.7$ and $1 \leqslant Re \leqslant 1000$, equations (7) and (8) correlate the values in Table 5 with a maximum error of 3.7% at $Re = 10$

$$N_T(\infty) = 0.04278 + 1.0414/Re, \quad \text{for } 1 \leqslant Re \leqslant 20 \tag{7}$$

$$N_T(\infty) = 0.06227 + 0.6980/Re, \quad \text{for } 20 \leqslant Re \leqslant 1000. \tag{8}$$

The dimensionless thermal entry length L^*_{th}, defined by $Nu_{x,T}(L^*_{th,T}) = 1.05Nu_T$ are given in Table 6. For simultaneously developing flow with $Pr = 0.7$, Shah and London suggest the value $L^*_{th,T} = 0.037$ based on the values of $Nu_{x,T}$ of both Manohar [10] and Hwang [11] in Table 3. This is 8% lower than the value of 0.0404 obtained in the present work. For the Graetz problem (uniform temperature profile at entrance and fully developed velocity profile), the thermal entrance length obtained in the present work (0.03333), see Nguyen [7], is very close to the analytic value (0.03347). The recent finite element analysis from Pagliarini gives values for $L^*_{th,T}$ which agree exactly with the present work (0.109 at $Re = 5$), but deteriorate to a 13% difference at higher Re (0.036 at $Re = 50$ and 0.035 at $Re = 500$).

Equations (9) and (10) approximate the values in Table 6 with a maximum error of 3.1% at $Re = 50$

$$L^*_{th,T} = 0.02460 + 0.4551/Re, \quad \text{for } 1 \leqslant Re \leqslant 20 \tag{9}$$

$$L^*_{th,T} = 0.03933 + 0.1767/Re, \quad \text{for } 20 \leqslant Re \leqslant 1000. \tag{10}$$

5. CONSTANT WALL HEAT FLUX RESULTS

Fully developed incremental heat transfer numbers for constant wall heat flux $N_H(\infty)$ are given in Table 7 for $1 \leqslant Re \leqslant 1000$ and $Pr = 0.7$. Equations (11) and (12) correlate the values in Table 7 with a maximum error of 2.1%

$$N_H(\infty) = 0.09058 + 0.2512/Re, \quad \text{for } 1 \leqslant Re \leqslant 20 \tag{11}$$

$$N_H(\infty) = 0.1026, \quad \text{for } 20 \leqslant Re \leqslant 1000. \tag{12}$$

For the hydrodynamically developed flow, Shah [18] obtained analytically the value of 0.04305 for $L^*_{th,H}$ which compares very well with the present numerical work (0.04290), see Nguyen [7]. The thermal entrance length $L^*_{th,H}$ for simultaneously developing flow and $Pr = 0.7$ is given in Shah and London as 0.053 and is in excellent agreement with the present work. The values of $L^*_{th,H}$ calculated by Pagliarini

Table 6. Thermal entrance length $L^*_{th,T}$ for $Pr = 0.7$

Re	1	2	5	10	20	50	100	200	1000
$L^*_{th,T}$	0.4787	0.2549	0.1090	0.06806	0.04873	0.04154	0.04066	0.04050	0.04040

Table 7. Incremental heat transfer number $N_H(\infty)$ for $Pr = 0.7$

Re	1	2	5	10	20	50	100	200	1000
$N_H(\infty)$	0.3414	0.2165	0.1430	0.1133	0.1034	0.1014	0.1022	0.1025	0.1026

Table 8. Thermal entrance length $L^*_{th,H}$

Re	1	2	5	10	20	50	100	200	1000
$L^*_{th,H}$	0.3296	0.1891	0.1059	0.07383	0.05942	0.05351	0.05268	0.05258	0.05257

for the case of constant heat flux agree very well with the values given in Table 8 for high Re (0.053 for $Re = 50$ and 0.052 for $Re \geqslant 100$). However, his value of 0.116 at $Re = 5$ is 8.7% higher than the present solution.

For $1 \leqslant Re \leqslant 1000$ and $Pr = 0.7$, equations (13) and (14) correlate the values of $L^*_{\mathrm{th,H}}$ presented in Table 8 with a maximum error of 2.4%

$$L^*_{\mathrm{th,H}} = 0.04663 + 0.2836/Re, \quad \text{for } 1 \leqslant Re \leqslant 20 \quad (13)$$

$$L^*_{\mathrm{th,H}} = 0.05163 + 0.1463/Re, \quad \text{for } 20 \leqslant Re \leqslant 1000. \quad (14)$$

6. CONCLUSION

The combined entry length problem in a circular tube with realistic upstream boundary conditions has been solved by a more accurate numerical method. Accurate Nusselt numbers, entrance lengths, and incremental heat transfer numbers are given for air and Re ranging from 1 to 1000. The results presented are found to correlate well with $1/Re$ for two distinct ranges of Re: low range $1 \leqslant Re \leqslant 20$; and medium to high range $20 \leqslant Re \leqslant 1000$. The present isothermal flow results are in excellent agreement with those obtained by Pagliarini and up to 16% higher than those by previous analyses. For non-isothermal flow, the present fully developed Nusselt numbers agree to five significant figures with analytic series solutions. The values of L^*_{th} given by Pagliarini for the simultaneously developing flow in a circular tube of infinite extent match the present solution only in parts of the Re range. No previous values of the incremental heat transfer numbers are available for comparison with the present work.

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